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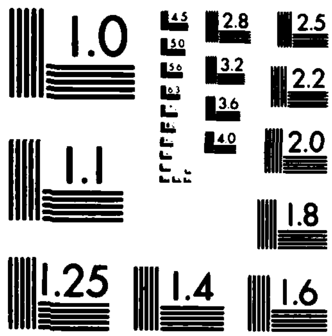
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ON THE ERGODICITY OF BILINEAR TIME SERIES MODELS AND SOME APPLICATIONS[1]

by

S. I. Akamanam[2], M. Bhaskara Rao and
K. Subramanyam

Center for Multivariate Analysis

University of Pittsburgh

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ON THE ERGODICITY OF BILINEAR TIME SERIES MODELS AND SOME APPLICATIONS

S. I. Akamanam, M. Bhaskara Rao, and K. Subramanyam

ABSTRACT

Existence, strict stationary and ergodicity of Bilinear Time Series Models for a given input White Noise process and parameter values is studied in detail in this paper. Using ergodicity of the model, estimation of the parameters by the method of moments is suggested and some comparisons are made with the method of least squares.

Key Words: Bilinear Time Series Models, Stationarity, Ergodicity, Method of moments and Method of Least Squares.

1. Introduction. Let X_t, $t \in Z$ and e_t, $t \in Z$ be two stochastic processes defined on some probability space $(\Omega, \mathcal{B}, P)$, where $Z = \{\ldots -1, 0, 1, \ldots\}$. X_t, $t \in Z$ is said to be a Bilinear Model with respect ot the process e_t, $t \in Z$ if

$$X_t = \sum_{j=1}^{r} a_j X_{t-j} + \sum_{j=1}^{h} b_j e_{t-j} + \sum_{\substack{i=1 \\ i \geq j}}^{m} \sum_{j=1}^{\ell} \beta_{ij} X_{t-i} e_{t-j} + e_t \quad \text{a.e.}[P] \text{ for every } t \text{ in } Z. \tag{1.1}$$

We designate e_t, $t \in Z$ to be the input or unobservable process and X_t, $t \in Z$ the output or observable process. The first part on the right side of (1.1) can be identified as the auto-regressive part of the process X_t, $t \in Z$, the second part as the moving average part of X_t, $t \in Z$ and the third part as the 'pure' bilinear part of X_t, $t \in Z$. A study of bilinear models, therefore, subsumes the study of auto-regressive, moving-average and mixed auto-regressive-moving-average models.

There are two basic questions that arise in this context.

(1) Suppose e_t, $t \in Z$; a_j's; b_j's and β_{ij}'s are all given. Under what conditions, is there a process X_t, $t \in Z$ satisfying (1.1)? We assume that e_t, $t \in Z$ is independent, identically distributed with $E\, e_t = 0$ and $E\, e_t^2 = \sigma^2 < \infty$.

(2) If a process X_t, $t \in Z$ exists satisfying (1.1), is it strictly stationary and also ergodic?

The study of bilinear models was initiated by Granger and Andersen [2] and Subba Rao [4]. The problem of existence of a process X_t, $t \in Z$ conforming to (1.1) has been studied by several authors in some special cases. Tuan Dinh Pham and Lanh Tat Tran [7] established the existence of a strictly stationary process X_t, $t \in Z$

satisfying

$$X_t = e_t + a X_{t-1} + b X_{t-1} e_{t-1} \quad \text{a.e.}[P]$$

for every $t \in Z$ under some conditions involving a,b and σ^2. Guegan [3] studied the existence problem of a process X_t, $t \in Z$ satisfying

$$X_t = e_t + b X_{t-2} e_{t-1} \quad \text{a.e.}[P]$$

for every t in Z. Subba Rao and Gabr [6] gave a set of sufficient conditions for the existence of a second order stationary process X_t, $t \in Z$ satisfying

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = e_t + \sum_{j=1}^{p} b_j X_{t-j} e_{t-j} \quad \text{a.e.}[P]$$

for every t in Z. Bhaskara Rao, Subba Rao and Walker [1] showed that under the same set of sufficient conditions given by Subba Rao and Gabr [6], there exists a strictly stationary process X_t, $t \in Z$ satisfying

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = e_j + \sum_{j=1}^{q} b_j X_{t-j} e_{t-j} \quad \text{a.e.}[P]$$

for every t in Z. This model is more general than the one considered by Subba Rao and Gabr [6] in the sense that p and q given above could be different. All these models are special cases of (1.1). The initial step in the proof of existence theorems given in Subba Rao and Gabr [6] and in Bhaskara Rao, Subba Rao and Walker [1] is to rewrite the above models in vectorial form. We pursue the same line of tack in studying the model (1.1).

In all the bilinear models studied in the literature, the moving average part has not been included. The purpose of this paper is to include the moving average part and also meet the following goals, using the method ginen in Bhaskara Rao, Subba Rao and Walker [1].

(1) Give a set of sufficient conditions under which there is a strictly stationary process X_t, $t \in Z$ satisfying (1.1). ((1.1) has moving average component.)

(2) Show that such a process is also ergodic.

(3) Show that such a process is unique

(4) Exploiting the ergodicity of the process, obtain estimators of the parameters of the process in a model fitting problem.

2. Vectorial representation.

We represent the model (1.1) in vector form. Suppose the processes X_t, $t \in Z$ and e_t, $t \in Z$ satisfy (1.1). Let

$$p = \max\{r,m\},$$

$$g = \min\{m,\ell\},$$

$$q = \max\{h,g\},$$

$$\underset{p\times p}{A} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_r & \overbrace{0 \quad 0 \quad \cdots \quad 0 \quad 0}^{(p-r)} \\ 1 & 0 & 0 & \cdots & 0 & 0 \quad 0 \quad \cdots \quad 0 \quad 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \quad 0 \qquad\;\; 0 \quad 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \quad 0 \quad \cdots \quad 1 \quad 0 \end{bmatrix},$$

$$\underset{p\times p}{C^T} = (1,0,0,\ldots,0),$$

$$\underset{1\times p}{\underline{b}_j^T} = (b_j,0,0,\ldots,0),\ j = 1,2,\ldots,h,$$

$$= \underline{0} \text{ for all } j = h+1,\ h+2,\ \ldots,q \text{ when } h < g,$$

$$\underset{p\times p}{B_j} = \begin{bmatrix} \beta_{jj} & \beta_{j+1,j} & \cdots & \beta_{mj} & 0 \quad 0 \quad \cdots \quad 0 \\ 0 & 0 & \cdots & 0 & 0 \quad 0 \quad \cdots \quad 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \quad \vdots \quad \cdots \quad \vdots \\ 0 & 0 & \cdots & 0 & \underbrace{0 \quad 0 \qquad\;\; 0}_{(p-m+j-1)} \end{bmatrix},$$

for $j = 1,2,\ldots,g$,

$$= \underline{0} \text{ for } j = g+1,\ g+2,\ldots,q \text{ when } h > g,$$

and

$$\underset{1\times p}{\underline{X}_t^T} = (X_t,\ X_{t-1},\ldots,x_{t-p+1}),\ t \in Z,$$

where $\underline{0}$ is the matrix of appropriate order in which every entry is zero and T stands for operation transpose of a matrix.

Theorem 1 If X_t, $t \in Z$ and e_t, $t \in Z$ satisfy (1.1), then

$$\underline{X}_t = C\, e_t + A\, \underline{X}_{t-1} + \sum_{j=1}^{q} \underline{b}_j\, e_{t-j} + \sum_{j=1}^{q} B_j\, \underline{X}_{t-j}\, e_{t-j} \quad \text{a.e.}[P] \qquad (2.1)$$

for every t in Z.

Proof. By direct verification.

We restate, in view of the above theorem, the existence problem as follows.

Let e_t, $t \in Z$ be a sequence of independent identically distributed random variables with $E\, e_t = 0$ and $E\, e_t^2 = \sigma^2 < \infty$. Let $\underset{p\times 1}{C}$, $\underset{p\times p}{A}$, $\underset{p\times 1}{\underline{b}_j}$, $j = 1,2,\ldots,q$ and B_j, $j = 1,2,\ldots,q$ are given matrices with real entries. Is there a strictly stationary process $\underset{p\times 1}{X_t}$, $t \in Z$ satisfying (2.1)?

3. Existence theorem

In this section, we show that under the condition stipulated in the main theorem of Bhaskara Rao, Subba Rao and Walker [1, p. 106], the strictly stationary process $\underline{X}_t$, $t \in Z$ *satisfying* (2.1) *not only* exists but also ergodic and essentially unique.

Theorem 2 Let e_t, $t \in Z$ be a sequence of independent identically distributed random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ with $E\, e_t = 0$ and $E\, e_t^2 = \sigma^2 < \infty$. Let $A, B_1, B_2, \ldots, B_q$ be $q + 1$ matrices of each of order of $p \times p$ and

$$\Gamma_1 = A \otimes A + \sigma^2 (B_1 \otimes B_1),$$

$$\Gamma_j = \sigma^2 [B_j \otimes (A^{j-1} B_1 + A^{j-2} B_2 + \ldots + A B_{j-1}) + (A^{j-1} B_1 + A^{j-2} B_2 + \ldots + A B_{j-1}) \otimes B_j + (B_j \otimes B_j)],$$

$$j = 2, 3, \ldots, q,$$

where $\otimes$ is the symbol for Kronecker product of matrices.

Suppose all the eigenvalues of the matrix

$$\underset{p^2 q \times p^2 q}{L} = \begin{bmatrix} \Gamma_1 & \Gamma_2 \ldots \Gamma_{q-1} & \Gamma_q \\ I_{p^2} & 0 \ldots 0 & 0 \\ 0 & I_{p^2} \ldots 0 & 0 \\ 0 & 0 \ldots I_{p^2} & 0 \end{bmatrix}$$

have moduli less than unity. Let $\underset{p\times 1}{C}$, $\underset{p\times 1}{\underline{b}_j}$, $j = 1, 2, \ldots q$ be given column vectors then there exists a strictly stationary and ergodic process $\underset{p\times 1}{\underline{X}_t}$, $t \in Z$ conforming to the model

$$\underline{X}_t = C\, e_t + A\, \underline{X}_{t-1} + \sum_{j=1}^{q} \underline{b}_j\, e_{t-j} + \sum_{j=1}^{q} B_j\, \underline{X}_{t-j}\, e_{t-j} \quad \text{a.e.}[P] \qquad (3.1)$$

for every $t \in Z$. Further, if a process U_t, $t \in Z$ conforms to the above bilinear model (3.1), then

$$U_t = \underline{X}_t \quad \text{a.e.}[P]$$

for every t in Z.

<u>Proof</u>. Let the process $\underline{S}_{n,t}$, n, t ε Z be defined as follows.

$$\begin{aligned}\underline{S}_{n,t} &= 0 \quad \text{if} \quad n < 0,\\ &= C\, e_t \text{ if } n = 0,\\ &= C\, e_t + \sum_{j=1}^{q} \underline{b}_j\, e_{t-j} + (A + B_1\, e_{t-1})\, S_{n-1,t-1}\\ &\qquad + B_2\, \underline{S}_{n-2,t-2}\, e_{t-2} + \ldots\\ &\qquad + B_q\, \underline{S}_{n-q,t-q}\, e_{t-q}, \text{ if } n > 0\end{aligned}$$

for every t in Z. We, first, observe that for each fixed n, $\underline{S}_{n,t}$, t ε Z is a strictly stationary process. Further, for each fixed n, $\underline{S}_{n,t}$ is measurable with respect to the σ-field $\sigma\{e_t, e_{t-1},\ldots\}$ for every t in Z, where $\sigma\{e_t, e_{t-1},\ldots\}$ is the smallest sub σ-field of $\mathcal{B}$ with respect to which each of e_t, $e_{t-1},\ldots$ is measurable. Consequently, the tail σ-field of $\{\underline{S}_{n,t}, t \in Z\}$=

$$\bigcup_{t=0}^{-\infty} \sigma\{\underline{S}_{n,t}, \underline{S}_{n,t-1},\ldots\} \subset \bigcup_{t=0}^{-\infty} \sigma\{e_t, e_{t-1},\ldots\} = \text{tail } \sigma\text{-field of}$$

$\{e_t, t \in Z\}$. Hence the process $\underline{S}_{n,t}$, t ε Z is ergodic. We show that $\lim_{n\to\infty} \underline{S}_{n,t}$ exists almost surely [P] for each fixed t in Z. If $\underline{X}_t$, t ε Z is the almost sure limit of $\underline{S}_{n,t}$, $n \geq 1$ for every t in Z, then it is obvious that the process $\underline{X}_t$, t ε Z conforms to the bilinear model (3.1). To show that $\underline{S}_{n,t}$ converges almost surely as $n \to \infty$ for every fixed t in Z, define

$$\begin{aligned}\underline{s}_{n,t} &= \underline{S}_{n,t} - \underline{S}_{n-1,t}\,, \quad n,\ t \in Z.\\ &= (A + B_1\, e_{t-1})\, \underline{s}_{n-1,t-1} + B_2\, \underline{s}_{n-2,t-2}\, e_{t-2}\\ &\quad +\ldots+ B_q\, \underline{s}_{n-q,t-q}\, e_{t-q}\end{aligned}$$

If we show that $E\,|(\underline{s}_{n,t})_i| \leq k\,\lambda^{n/2}$, it follows that $\underline{S}_{n,t}$, $n \geq 1$ converges a.e.[P] for every $t \in Z$. See Step 2° in Bhaskara Rao, Subba Rao and Walker [1, p. 106], where λ is the maximum absolute eigenvalue of L, $(\underline{s}_{n,t})_i$ is the i^{th} - component of $\underline{s}_{n,t}$ and k is some positive constant. Following Steps 3° to 10° in Bhaskara Rao, Subba Rao and Walker [1], one can show that $E\,|(\underline{s}_{n,t})_i| \leq k\,\lambda^{n/2}$ for every n and i = 1,2,...,0.

The limit process $\underline{X}_t$, $t \in Z$ is also ergodic. This follows from the fact that $\underline{X}_t$ is measurable with respect to $\sigma\{e_t, e_{t-1}, \ldots\}$ for every t in Z.

Now, we prove uniqueness. Let $\underline{U}_t$, $t \in Z$ be a process conforming to (3.1). For each $n \geq q$ and $t \in Z$, let

$$\underline{V}_{n,t} = \underline{U}_t - \underline{S}_{n,t}.$$

Then
$$\underline{V}_{n,t} = (A + B_1\, e_{t-1})\, \underline{V}_{n-1,t-1} + B_2\, \underline{V}_{n-2,t-2} + \ldots + B_q\, \underline{V}_{n-q,t-q}.$$

As above, one can show that $E\,|(\underline{V}_{n,t})_i| < k\,\lambda^{n/2}$ for every n, t and i. Since $\lambda < 1$, it follows that $\lim_{n\to\infty} \underline{V}_{n,t} = 0$ a.e.[P]. But

$$\lim_{n\to\infty} \underline{V}_{n,t} = \lim_{n\to\infty} (\underline{U}_t - \underline{S}_{n,t})$$
$$= \underline{U}_t - X_t = 0 \quad \text{a.e.[P]}.$$

This completes the proof of uniqueness part of the theorem.

<u>Remarks</u>.

(1) Theorem 2 remains valid if we have a moving average part of general order not necessarily of order q same as the order of the pure bilinear part.

(2) The most important feature that emerges by comparing Theorem 2 above and the main theorem in Section 4 of Bhaskara Rao,

Subba Rao and Walker [1, p. 106] is that the presence of moving average part makes no impact on the existence problem. This is also typical of Linear processes as the following corollary shows.

Corollary 3 Let e_t, $t \in Z$ be a sequence of independent identically distributed real random variables with common mean 0 and variance $\sigma^2 < \infty$. Then there exists a strictly stationary process X_t, $t \in Z$ satisfying

$$X_t = \sum_{j=1}^{r} a_j X_{t-j} + \sum_{j=1}^{\ell} b_j e_{t-j} + e_t \quad \text{a.e.}[P] \qquad (3.2)$$

for every t in Z if the roots of the polynomial

$$f(x) = 1 - a_1 x - a_2 x^2 - \ldots - a_r x^r$$

are in absolute value greater then unity.

Proof. The model (3.2) can be put in the vector form as follows.

Let $p = r$, $q = \ell$,

$$\underset{p\times p}{A} = \begin{bmatrix} a_1 & a_2 & a_3 \ldots & a_{p-1} & a_p \\ 1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 1 & 0 \ldots & 0 & 0 \\ \vdots & \vdots & \vdots \ldots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & 1 & 0 \end{bmatrix},$$

$$\underset{1\times p}{\underline{b}_j^T} = (b_j,0,0,\ldots,0),\ j = 1,2,\ldots,q,$$

$$\underset{1\times p}{C^T} = (1,0,0,\ldots 0), \quad \text{and}$$

$$\underset{1\times p}{\underline{X}_t^T} = (X_t, X_{t-1}, \ldots, X_{t-p+1}),\ t \in Z.$$

Then

$$\underline{X}_t = C\, e_{t-1} + A\, \underline{X}_{t-1} + \sum_{j=1}^{q} \underline{b}_j\, e_{t-j} \quad \text{a.e.}[P] \qquad (3.3)$$

for every t in Z.

A sufficient condition for the existence of a strictly stationary real vector-vlaued process $\underline{X}_t$, $t \varepsilon Z$ satisfying (3.3) is that the maximum absolute eigenvalue of $A \otimes A$ is less than unity. This condition is equivalent to the condition that the maximum absolute eigenvalue of A is less than unity. This in turn is equivalent to the condition that the roots of the characteristic polynomial are less than unity in modulus. This completes the proof.

4. On estimation of the parameters of a bilinear model

Suppose X_t, $t \in Z$ is a real valued stochastic process satisfying

$$X_t = e_t + \sum_{j=1}^{r} b_j e_{t-j} + a X_{t-1} + \sum_{j=1}^{q} a_j X_{t-j} e_{t-j} \quad \text{a.e.}[P]$$

for every t in Z for some sequence e_t, $t \in Z$ of independent identically distributed random variables with $E e_t = 0$ and $E e_t^m < \infty$ for sufficiently large m. There are $q + r + 1$ parameters $a, a_1, a_2, \ldots, a_2, b_1, b_2, \ldots, b_r$ in the above model which we want to estimate based on realizations $x_1, x_2, \ldots, x_N$ of $X_1, X_2, \ldots, X_N$ respectively. Assuming the process X_t, $t \in Z$ to be strictly stationary and ergodic, we proceed with the problem of estimation as follows. (Assume, for simplicity, that the process e_t, $t \in Z$ is completely specified.)

Step 1 Calculate $E X_t, E X_t^2, \ldots, E X_t^{q+r+1}$.

Step 2 Let $E X_t^s = f_s(a, a_1, a_2, \ldots, a_q, b_1, b_2, \ldots, b_r)$,

$s = 1,2,\ldots,q+r+1.$

Step 3 Estimates based on the Method of Moments.

Solve the equations

$$f_s(\hat{a}, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_q, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) = \frac{1}{N} \sum_{i=1}^{N} x_t^s,$$

$s = 1,2,\ldots,q+r+1$

in the unknowns $\hat{a}, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_q, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r$.

This method is justified on the grounds that the process X_t^s, $t \in Z$ is strictly stationary and ergodic for $s = 1,2,3,\ldots$ and that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X_t^s = E X_t^s \quad \text{a.e.}[P]$$

for every $s \geq 1$.

An example

Let e_t, $t \in Z$ be a sequence of independent identically distributed random variables with e_t having normal distribution with mean 0 and variance unity. Let X_t, $t \in Z$ be the strictly stationary and ergodic process satisfying

$$X_t = e_t + a\,X_{t-1} + b\,X_{t-1}\,e_{t-1} \quad \text{a.e.}[P]$$

for every t in Z with a = b = 0.4. It can be checked that $E\,X_t = \frac{b}{1-a}$ and $E\,X_t^2 = (1 / 1 - a^2 - b^2)\,(1 + 2\,b^2(1+a) / (1-a))$.

We have generated 1000 observations $x_1, x_2, \ldots, x_{1000}$ from the above model. The estimates $\hat{a}$ and $\hat{b}$ of a and b respectively based on the method of moments are obtained by solving the equations

$$\frac{\hat{a}}{1-\hat{b}} = \frac{1}{1000} \sum_{i=1}^{1000} x_i = 0.5854906$$

and

$$\frac{1}{1-\hat{a}^2-\hat{b}^2} \left\{ 1 + \frac{2\,\hat{b}^2(1+\hat{a})}{1-\hat{a}} \right\} = \frac{1}{1000} \sum_{i=1}^{1000} x_i^2 = 2.197616$$

in the unknowns $\hat{a}$ and $\hat{b}$. The solution is given by $\hat{a} = 0.3963$ and $\hat{b} = 0.4281$.

Following Subba Rao [4],[5], one can apply the method of least squares to minimize $Q(a,b) = \sum_t e_t^2$ over a,b, where $e_t = x_t - a\,x_{t-1} - b\,x_{t-1}\,e_{t-1}$ $t = 1,2,\ldots,1000$, to obtain the following recursive equations.

$$\frac{\partial e_t}{\partial a} + b\,x_{t-1} \frac{\partial e_{t-1}}{\partial a} = -x_{t-1},$$

$$\frac{\partial e_t}{\partial b} + b\,x_{t-1} \frac{\partial e_{t-1}}{\partial b} = -x_{t-1}\,e_{t-1},$$

$$\frac{\partial^2 e_t}{\partial a^2} + b\,x_{t-1} \frac{\partial^2 e_{t-1}}{\partial a^2} = 0,$$

$$\frac{\partial^2 e_t}{\partial b^2} + b\,x_{t-1} \frac{\partial^2 e_{t-1}}{\partial b^2} + 2\,x_{t-1} \frac{\partial e_{t-1}}{\partial b} = 0,$$

and

$$\frac{\partial^2 e_t}{\partial a \partial b} + x_{t-1} \frac{\partial e_{t-1}}{\partial a} + b\,x_{t-1} \frac{\partial^2 e_{t-1}}{\partial a \partial b} = 0,$$

$t = 1,2,\ldots,1000.$

Let $G^T(a,b) = \left(\frac{\partial Q(a,b)}{\partial a}, \frac{\partial Q(a,b)}{\partial b} \right)$ and

$$H(a,b) = \begin{pmatrix} \frac{\partial^2 Q(a,b)}{\partial a^2} & \frac{\partial^2 Q(a,b)}{\partial a \partial b} \\ \frac{\partial^2 Q(a,b)}{\partial a \partial b} & \frac{\partial^2 Q(a,b)}{\partial b^2} \end{pmatrix},$$

where one can easily check that

$$\frac{\partial Q(a,b)}{\partial a} = 2 \sum_{t=1}^{1000} e_t \frac{\partial e_t}{\partial a},$$

$$\frac{\partial Q(a,b)}{\partial b} = 2 \sum_{t=1}^{1000} e_t \frac{\partial e_t}{\partial b},$$

$$\frac{\partial^2 Q(a,b)}{\partial^2 a} = 2 \sum_{t=1}^{1000} \left(\frac{\partial e_t}{\partial a} \right)^2 + 2 \sum_{t=1}^{1000} e_t \frac{\partial^2 e_t}{\partial a^2},$$

$$\frac{\partial^2 Q(a,b)}{\partial^2 b} = 2 \sum_{t=1}^{1000} \left(\frac{\partial e_t}{\partial b} \right)^2 + 2 \sum_{t=1}^{1000} e_t \frac{\partial^2 e_t}{\partial b^2}$$

and

$$\frac{\partial^2 Q(a,b)}{\partial a \partial b} = 2 \sum_{t=1}^{1000} \left(\frac{\partial e_t}{\partial a} \right) \left(\frac{\partial e_t}{\partial b} \right) + 2 \sum_{t=1}^{1000} e_t \frac{\partial^2 e_t}{\partial a \partial b}$$

Starting with initial values $a^{(0)}$, $b^{(0)}$ and

$x_0 = \frac{\partial e_0}{\partial a} = \frac{\partial e_0}{\partial b} = \frac{\partial^2 e_0}{\partial a^2} = \frac{\partial^2 e_0}{\partial b^2} = \frac{\partial^2 e_0}{\partial a \partial b} = 0$, we obtain the Newton-Raphson iterative equation

$$\begin{pmatrix} a^{(k+1)} \\ b^{(k+1)} \end{pmatrix} = \begin{pmatrix} a^{(k)} \\ b^{(k)} \end{pmatrix} - [H(a^{(k)}, b^{(k)})]^{-1} G(a^{(k)}, b^{(k)}).$$

Starting with $a^{(0)} = b^{(0)} = 0.1$, using the recursive equations above, the Newton-Raphson method gives

$$\hat{a} = 0.3874$$
$$\hat{b} = 0.3958$$

correct to 4 decimal places. These values are very close to the true parameter values $a = b = 0.4$. Estimates obtained by the method of moments do not come as close to the true parameter values as those obtained by the method of least squares for smaller samples. But the method of moments has computational simplicity that the method of least squares lacks. One could use the estimates given by the method of moments as the starting-up values for the method of least squares cranking up the Newton-Raphson machinery in small samples. However, if the sample size is large, both methods give values very close to the true parameter values.

Concluding Remarks

We have shown that under some simple condition in the spectral radius of a matrix, Bilinear models do exist, are stationary and ergodic. Ergodicity of the process makes the method of moments as a natural technique for adoption to estimate the parameters of the model. This method is compared with the usual method of least squares and found to be satisfactory in large samples.

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